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On gravitation theories with limiting curvature

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Abstract. We discuss the solutions of the generalised gravitational field equations derived from the nonlinear Lagrangian for a limiting curvature theory and compare the free space solutions as well as the solutions for an extended star with the corresponding solutions of the Einstein equations.

1. Introduction

One of the curiosities of Einstein's theory of gravitation is the appearance of event horizons and singularities. An *event horizon* is characterised by a wave front (null hypersurface) which just cannot escape to infinity. In the classical theory it is penetrable only in one direction and thus generates a strong limitation in the causal relationship between different parts of space–time. In contrast, an *invariant singularity* is a point in a space–time manifold at which the normal picture of space–time breaks down, caused, e.g., by an infinite Riemannian curvature in the vicinity of a point mass. It has been shown in a number of theorems that under very general conditions singularities and horizons appear in Einstein's theory. This circumstance may speak against Einstein's Lagrangian among competitive theories. On the other hand, in the last few years, a number of experiments have proven many of Einstein's predictions to an accuracy of about 1%.

This failure of the classical theory to avoid singularities naturally leads to the question whether this is remedied in a quantised theory. Here one has to distinguish two steps: (1) quantisation of the matter fields $T_{\mu\nu}$ coupling to the classical (external) gravitational field; (2) quantisation of the gravitational field itself. The second scheme is the more challenging one, but seems to be outside the present reach since no generally accepted method of quantising the gravitational field is available. However, it deserves thorough investigation whether or not the quantisation of matter and radiation fields already changes the situation.

Recent theoretical work indicates that vacuum polarisation and particle creation play an important role in strong gravitation fields (Hawking 1974, 1975, DeWitt 1975, Christensen 1976, Rumpf 1976a,b, Christensen and Fulling 1977 and Soffel *et al* 1977). In this paper, we shall try to treat vacuum polarisation caused by a strong gravitational field in a phenomenological way by including nonlinear terms in the Lagrangian for the gravitational field. When doing so, we are guided by the result of Heisenberg and Euler (1936) who showed that vacuum polarisation in strong electromagnetic fields can be effectively described by nonlinearities in the Lagrangian for the electromagnetic interaction. Born and Infeld (1934) added to this the idea of a limiting electromagnetic field strength thus eliminating the divergences of Maxwell's theory. In this spirit we shall postulate the appearance of a limiting curvature scalar R_0, which should form an upper limit to space-time curvature. Such a theory will avoid from the beginning the existence of space-time singularities caused by infinite Riemannian curvature. The question, whether such a modification of the Lagrangian will do away with the appearance of event horizons in gravitational collapse as well, will be discussed in the last section of this paper.

2. The model Lagrangian and the field equations

The field equations for an arbitrary Lagrangian density $\mathscr{L}=\sqrt{-g}f(R)$ are obtained through the variation of the action integral

$$J=\int \mathscr{L}\,\mathrm{d}^4x=\int \mathrm{d}^4x\sqrt{-g}f(R) \tag{1}$$

with respect to the metric tensor g_{ik}. The variation of J is straightforward (see e.g. Lanczos 1925, 1932 for relations between δR_{ik} and δg^{ik}). The resulting field equations read:

$$\begin{aligned}H_{ik}\equiv f^{(3)}(R)(R_{|i}R_{|k}-g_{ik}R_{|s}R_{|r}g^{rs})\\ +f^{(2)}(R)(R_{|k\|i}-g_{ik}\Box R)+f'(R)R_{ik}-\tfrac{1}{2}f(R)g_{ik}=0,\end{aligned} \tag{2}$$

where f', $f^{(2)}$ etc denote derivatives with respect to R, and $|$ and $\|$ denote ordinary and covariant derivatives with respect to the coordinates, respectively. The field equations are divergenceless as can be shown by explicit covariant differentiation of $H^i{}_k$. Already it can be noted that equations (2) depend on derivatives of the metric tensor g_{ik} up to fourth order. The consequences of this unusual feature will show up in the following.

The Lagrangian is to be modelled after existing Lagrangians of other limiting field theories, i.e. Born–Infeld theory as discussed by Rafelski *et al* (1972), or the Lagrangian of the theory of special relativity. We have chosen the following Lagrangian density:

$$\begin{aligned}&\mathscr{L}=f(R)\sqrt{-g}\\ &f(R)=-\frac{R_0}{n}\left[\left(1-\frac{R}{R_0}\right)^n-1\right] \qquad (n<1).\end{aligned} \tag{3}$$

It fulfills several important conditions one would want to impose on every physically reasonable theory. In regions of vanishing curvature the Lagrangian goes over into the Lagrangian of general relativity; i.e. $f(R)/R\to 1$ for $R\to 0$. In addition, the existence of an R^2 term in $f(R)$ ensures the possibility of finding asymptotically flat solutions. This can be seen by forming the trace of equations (2) and taking $g^{00}\sim 1$, $g^{11}\sim -1$ and $R=R(r)$:

$$\frac{\partial^2}{\partial r^2}(rf'(R))+\tfrac{1}{3}r(Rf'(R)-2f(R))=0. \tag{4}$$

For $f(R)=R+\xi R^{\gamma}$ one obtains the asymptotic solutions

$$R\sim\begin{cases} cr^{\gamma/(\gamma-1)(\gamma-2)} & (\gamma\neq 2) \quad (5a)\\ \dfrac{c}{r}\exp[\pm r/\sqrt{(6\xi)}] & (\gamma=2). \quad (5b)\end{cases}$$

We see that we must have $\gamma=2$ necessarily and $\xi>0$ to make R vanish at large distances from the source. The last condition requires $n<1$. It is clear from dimensional arguments that the limiting curvature R_0 can be related to a 'limiting' mass density T_0 by the formula

$$R_0=\frac{8\pi\kappa}{c^2}T_0.$$

In the absence of experimental results this relation can be used to obtain a theoretical guess of the magnitude of R_0. If we set $c^2\,T_0\cong 1$ GeV fm^{-3}, which corresponds to six times the normal nuclear density, one obtains

$$R_0\approx(60\text{ km})^{-2}.$$

This defines a characteristic length, which is equal to the Schwarzschild radius of a star of about twenty solar masses.

3. Solution of the field equations in empty space

We use the metric $g_{\mu\nu}=\text{diag}(e^{\nu(r)},-e^{\sigma(r)},-r^2,-r^2\sin^2\theta)$. Under the assumption of radial symmetry and time-independence the 16 equations (2) reduce to the following three equations of the two independent variables $\nu(r)$ and $\sigma(r)$:

$$R''=\frac{n-2}{R_0}\left(1-\frac{R}{R_0}\right)^{-1}R'^2+R'\left(\frac{\sigma'}{2}-\frac{2}{r}\right)-\frac{R_0}{n-1}\left(1-\frac{R}{R_0}\right)\left(-\frac{e^{\sigma}}{2}R+\frac{\sigma'}{r}-\frac{1}{r^2}+\frac{e^{\sigma}}{r^2}\right)$$
$$+\frac{e^{\sigma}}{2}\frac{R_0^2}{n(n-1)}\left[\left(1-\frac{R}{R_0}\right)^n-1\right]\left(1-\frac{R}{R_0}\right)^{2-n} \quad (6a)$$

$$\nu'=\left[1-\frac{n-1}{R_0}\left(1-\frac{R}{R_0}\right)^{-1}\frac{rR'}{2}\right]^{-1}\left\{\frac{n-1}{R_0}\left(1-\frac{R}{R_0}\right)^{-1}2R'-\frac{e^{\sigma}}{2}rR-\frac{1}{r}\right.$$
$$\left.+\frac{e^{\sigma}}{r}-\frac{re^{\sigma}}{2}\frac{R_0}{n}\left[\left(1-\frac{R}{R_0}\right)-\left(1-\frac{R}{R_0}\right)^{1-n}\right]\right\} \quad (6b)$$

$$\sigma'=-\frac{n-1}{R_0}\left(1-\frac{R}{R_0}\right)^{-1}\left(\frac{R'}{3}(-\nu'r+2)\right)+\frac{1}{3}\left[\nu'+\frac{4}{r}+e^{\sigma}\left(Rr-\frac{4}{r}\right)\right]. \quad (6c)$$

Of these three equations only two are independent due to the relation $H^i{}_{k\|i}=0$.

The field equations allow only three types of solutions: (*a*) $R\equiv 0$; (*b*) $R\equiv g(n)R_0=\text{constant}$, where $g(n)$ depends on the parameter n in the Lagrangian and $g(n)=\frac{8}{9}$ for $n=\frac{1}{2}$; (*c*) $R(r)\to R_0$ for $r\to 0$. The first type of solution includes the flat space and the Schwarzschild solution. The latter, however, is not a proper solution when the nature of the point source is considered explicitly (see § 6 and appendix). The second type of solution does not approach flat space at infinity and thus might only be of cosmological interest. It contains the solution of the Einstein equations with the cosmological Λ-term for $\Lambda = \frac{2}{9}R_0$ in the case $n = \frac{1}{2}$. These types obviously can not correspond to solutions for an isolated star, leaving only case (c) as a possible solution. Therefore we shall generally assume $R \to R_0$ at the origin for the point-source solutions.

In order to find solutions $\nu(r)$, $\sigma(r)$, $R(r)$ of the field equations and to be able to impose proper boundary conditions, we first have to study the asymptotic behaviour for $r \to \infty$ and $r \to 0$.

3.1. *Solution for large r*

At large distances an *ansatz* for the metric e^{σ} and e^{ν} was obtained by considering a small deviation from the Schwarzschild solution e^{ν_s}, e^{σ_s},

$$e^{\nu} = e^{\nu_s} + \epsilon(r) \qquad (7)$$

$$e^{-\sigma} = e^{-\sigma_s} + \eta(r) \qquad (\nu_s = -\sigma_s)$$

where ϵ, η are small for large r. Inserting (7) into the field equations (6) and keeping terms linear in ϵ and η only, one is led to the following asymptotic expansion of the metric ($m = \kappa M/c^2$, where M is the mass of the star):

$$e^{\nu} = 1 - \frac{2m}{r} + \alpha r^{-(m/a)-1}\, e^{-r/a}[1 + O(r^{-1})] \qquad (8)$$

$$e^{-\sigma} = 1 - \frac{2m}{r} - \frac{\alpha}{a} r^{-m/a}\, e^{-r/a}[1 + O(r^{-1})] \qquad (9)$$

with an arbitrary constant α and

$$a^2 = 3(1-n)/R_0. \qquad (10)$$

Independently, from the contracted form of equations (2), and imposing the boundary condition that R must vanish as $r \to \infty$ one obtains

$$R \approx -\frac{3\alpha}{a^2} r^{-(m/a)-1}\, e^{-r/a}[1 + O(r^{-1})]. \qquad (11)$$

3.2. *Solution for small r*

An appropriate solution in the vicinity of $r = 0$ has been found via Laurent expansion of ν' and σ' around $r = 0$:

$$\nu' = \sum_{-\infty}^{\infty} \nu_i r^i \qquad (12a)$$

$$\sigma' = \sum_{-\infty}^{\infty} \sigma_k r^k. \qquad (12b)$$

If we require that

$$\lim_{r\to 0} R = \text{constant} \leq R_0 \qquad (13)$$

to ensure the reality of the Lagrangian L, we find that

$$\nu_i = 0, \qquad \sigma_k = 0 \qquad \text{for all } i, k < -1. \qquad (14)$$

To obtain the nonzero coefficients ν_i, σ_k we solve the field equations in the vicinity of $r=0$. With the help of the field equations (6*a*)–(6*c*) and the curvature scalar

$$R=-\mathrm{e}^{-\sigma}\left(\nu''+\frac{\nu'^2}{2}+\frac{2}{r}(\nu'-\sigma')-\frac{\nu'\sigma'}{2}+\frac{2}{r^2}\right)+\frac{2}{r^2}, \tag{15}$$

we obtain the relations

$$\sigma_{-1}=\frac{\nu_{-1}(\nu_{-1}+2)+4}{\nu_{-1}+4} \tag{16}$$

and

$$R=R_0-Ar^\rho+0(r^{\rho+1}) \qquad \text{with } \rho=\frac{2(\nu_{-1}+1)}{(1-n)(\nu_{-1}+4)}. \tag{17}$$

In order that the divergent term $2/r^2$ in the expression for R in (15) can be cancelled, σ_{-1} must be integer and greater than zero. For ν_{-1} all corresponding values greater than -1 are allowed. $\sigma_{-1}=1$, $\nu_{-1}=-1$ again leads to the Schwarschild solution and is excluded.

3.3. *Global numerical solutions*

As physical constraints we impose that the metric be finite everywhere. Also it must have the Newtonian limit at large distances. The differential equations are of fourth order, hence we have to impose an additional constraint. We require that the velocity of light stays limited as r goes to zero. Variation of the parameter α then leads to solutions which in all calculations gave

$$\lim_{r\to 0} R = R_0$$

where R_0 is the limiting curvature (equation (3)). For integration of equations (6*a*, *b*, *c*) we used a logarithmic distribution of integration points. We employed the predictor–corrector method based on the Adams formula (Abramowitz and Stegun 1964). For starting values we took the asymptotic expansion of equations (8), (9). The numerical calculations proved $\sigma_{-1}=\nu_{-1}=2$ to be the only solution satisfying the physically reasonable condition that the speed of light should stay finite as r approaches zero. So the metric coefficients are

$$\mathrm{e}^{\sigma}=(r/r_0)^2[1+\sigma_1 r+r^2\tfrac{1}{2}(\sigma_2+\sigma_1^2)+\ldots], \tag{18a}$$

$$\mathrm{e}^{\nu}=(r/r_0')^2[1+\sigma_1 r+r^2(\tfrac{3}{2}\sigma_2+1/r_0^2+\tfrac{1}{2}\sigma_1^2)+\ldots], \tag{18b}$$

where r_0 and r_0' are arbitrary constants related to σ_0 and ν_0 in the expansion of $\sigma(r)$ and $\nu(r)$.

The parameters entering into our theory can be expressed in dimensions of length, including the limiting curvature R_0. Since there is no absolute unit of length the theory must be scale invariant. Indeed, all results remain invariant under the simultaneous transformation

$$(r, m, a, \alpha)\to(\lambda r, \lambda m, \lambda a, \lambda\alpha) \qquad \text{and} \qquad R\to\lambda^{-2}R. \tag{19}$$

Therefore all results are shown in arbitrary units of r. Results obtained under these assumptions are shown in figure 1 for a variation of the parameter a, i.e. varying

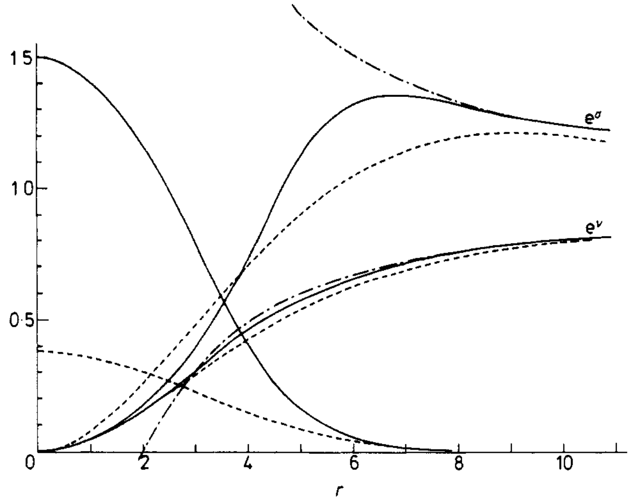

Figure 1. The metric components e^ν, e^σ and the curvature scalar $R(r)$ are plotted for the variation of the limiting curvature R_0 using the parameter $a=[3(1-n)/R_0]^{1/2}$ (full curves, $a=1\cdot0$; broken curves, $a=2\cdot0$). The Schwarschild solution ($a=0$) is shown for comparison (chain curves). Observe that no event horizon arises in the limiting curvature theory. $n=0\cdot5$ and $m=1\cdot0$.

curvature limit R_0. *There is no singularity left at $r=2m$ nor does there appear any pathological behaviour at any other distance. Both e^σ and e^ν keep the same signature as in the outside world of the Schwarzschild solution and go to zero for $r\to0$.* This means that t is a time-like and r a space-like coordinate in all regions of space-time. e^σ shows a maximum moving outward as the limiting curvature R_0 decreases, whereas e^ν changes only slightly under variation of R_0.

Qualitatively the same behaviour appears in the variation of the mass m and the exponent of the Lagrangian density n (figures 2 and 3). Again, e^σ has a flat maximum which is shifting outward for increasing central mass. At the same time the width of the curve $R(r)$ grows. In figure 3, we have shown only $R(r)$ for three different exponents n in the Lagrange function (equation (3)). For $n\leqslant-1$ the curvature scalar does not have a vanishing slope at the origin, which can be understood from formula (17). The exponential tail of R, however, remains unchanged.

Further, in figure 4, we have plotted the apparent (for an asymptotic observer) velocity of light for various values of R_0. In accordance with the positive-definiteness of the metric components, we find that $v_{\text{light}}=c\exp[\frac{1}{2}(\nu-\sigma)]$ is finite everywhere and smaller than c. Thus information from all points can be propagated to infinity and no event horizon is formed.

In the following we briefly discuss the consequences of the limiting curvature theory with respect to the three classical tests on general relativity, i.e. gravitational redshift, perihelion shift and the deflection of starlight passing the sun. Since all tests are performed in weak gravitational fields one can use the parametrised post-Newtonian formalism to discuss the consequences. In all three cases a detailed analysis (Müller A, 1977) shows that the deviations from general relativity are of the order of

$$e^\nu-e^{\nu_s}\sim e^{-r/a} \qquad \text{or} \qquad e^\sigma-e^{\sigma_s}\sim e^{-r/a}.$$

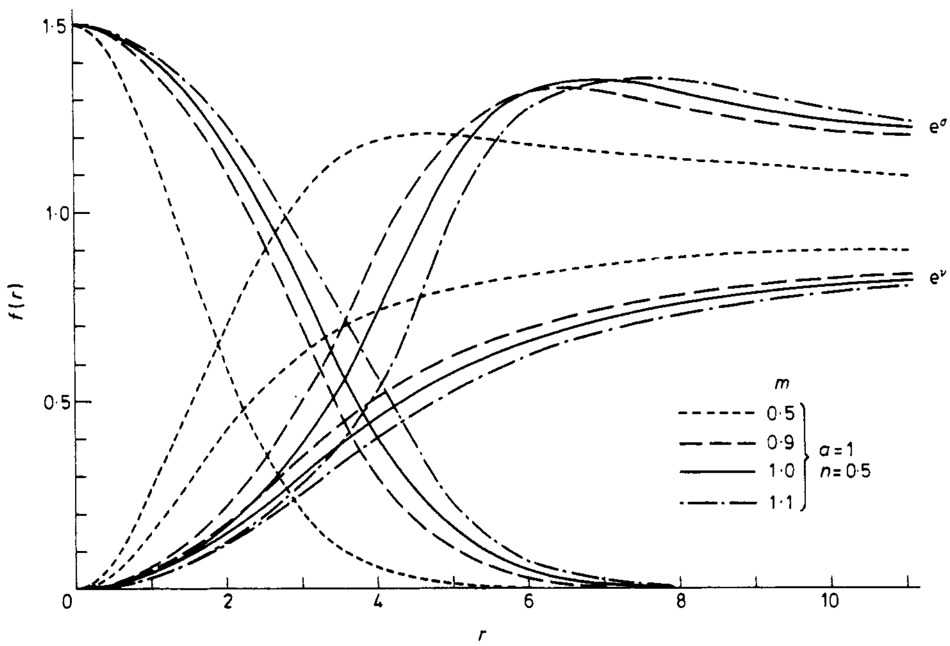

Figure 2. The variations of the metric components e^{σ}, e^{ν} and the curvature scalar R are shown as a function of the mass m of the source of the gravitational field.

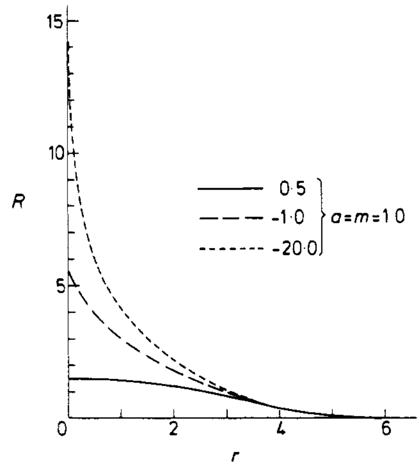

Figure 3. The curvature scalar $R(r)$ is shown for several values of the exponent n in the Lagrange function. The influence on the components of the metric tensor is too small to be depicted.

As a should be less than 100 km, possibly even smaller, and all experiments are performed at $r \gg 10\,000$ km, the deviations do not contradict any present or forseeable precision experiment. Strong effects can only be expected close to the surface of small compact stellar objects, e.g. neutron stars.

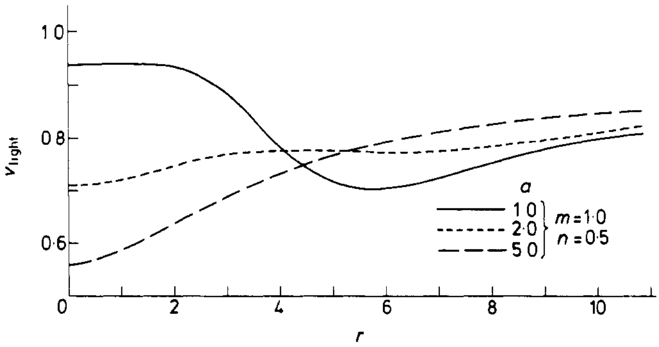

Figure 4. This figure shows the velocity of light as seen by a distant observer. It stays finite everywhere, so that no event horizon occurs.

Finally let us explore the tidal forces present in our solution (18*a*, *b*). The curvature invariant

$$R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}=\frac{e^{2\nu}}{16}(\sigma'\nu'-2\nu''-\nu'^2)^2+\frac{\nu'^2}{2r^2}e^{-2\sigma}+\frac{\sigma'^2}{2r^2}e^{-2\sigma}+\frac{(1-e^{-\sigma})^2}{r^4} \tag{20}$$

for the metric (18) becomes

$$R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}=\frac{1}{(r_0')^4}+\frac{4r_0^4}{r^8}+\frac{1}{r^4}\left[1-\left(\frac{r_0}{r}\right)^2\right]^2 \tag{21}$$

with the numerically defined constants r_0 and r_0'. Here, obviously, the first term resulting exclusively from the radial component $R^{01}{}_{01}$ stays finite, while in the case of the Schwarzschild solution the whole expression diverges at the origin:

$$R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}=48m^2/r^6. \tag{22}$$

To understand the physical meaning of this result, we have to remind the reader of the role of the $R_{\mu\nu\sigma\rho}$ as determinants of goedesic deviation: the relative acceleration between bodies on neighbouring geodesics is given by $\partial^2\xi^k/\partial\tau^2=-R^{0k}{}_{0i}\xi^i$, where $\xi^k(\tau)$ is the shortest distance in between and τ is the proper time on the geodesic. The radial component $R^{01}{}_{01}$ stays finite, i.e. there exists no infinite radial tidal pressure as in the Schwarzschild solution at $r\to 0$. The diverging of the angular tidal forces is inherent in the model of a point mass, i.e. is forced upon the solution from the beginning. Whether this will happen practically, depends on whether gravitational collapse to a point mass still occurs in our theory. For this, however, we have to solve the field equations for an extended star.

4. Solution of the field equations for a star consisting of a perfect fluid

In order to come to a better understanding of the behaviour of the solutions for empty space–time, we solved the inhomogeneous equations (see equation (2))

$$H_{ik}=-\frac{8\pi\kappa}{c^2}T_{ik} \tag{23}$$

for a spherical star consisting of an incompressible fluid with density ρ_0:

$$T^i{}_k = \mathrm{diag}\left(\rho_0, -\frac{p}{c^2}, -\frac{p}{c^2}, -\frac{p}{c^2}\right). \tag{24}$$

In this case, using the spherically symmetric and time-independent metric of § 4, the three equations (6) are amended by the hydrostatic equation for the pressure $p(r)$:

$$R'' = \frac{n-2}{R_0}\left(1-\frac{R}{R_0}\right)^{-1} R'^2 + R'\left(\frac{\sigma'}{2}-\frac{2}{r}\right) - \frac{R_0}{n-1}\left(1-\frac{R}{R_0}\right)\left(-\frac{e^\sigma}{2}R + \frac{\sigma'}{r} - \frac{1}{r^2} + \frac{e^\sigma}{r^2}\right)$$
$$+\frac{e^\sigma}{2}\frac{R_0^2}{n(n-1)}\left[\left(1-\frac{R}{R_0}\right)^2 - \left(1-\frac{R}{R_0}\right)^{2-n}\right] + \frac{R_0}{n-1} e^\sigma \left(1-\frac{R}{R_0}\right)^{2-n} \frac{8\pi\kappa}{c^2}\rho_0 \tag{25a}$$

$$\nu' = \left\{2\frac{n-1}{R_0}R' + \left(1-\frac{R}{R_0}\right)\left(-\frac{re^\sigma}{2} - \frac{1}{r} + \frac{e^\sigma}{r}\right) - \frac{re^\sigma}{2}\frac{R_0}{n}\left[\left(1-\frac{R}{R_0}\right)^2 - \left(1-\frac{R}{R_0}\right)^{2-n}\right]\right.$$
$$\left. + re^\sigma\left(1-\frac{R}{R_0}\right)^{2-n}\frac{8\pi\kappa}{c^4}p\right\}\left[-\frac{r}{2}\frac{n-1}{R_0}R' + \left(1-\frac{R}{R_0}\right)\right]^{-1} \tag{25b}$$

$$3\sigma' = -\frac{n-1}{R_0}\left(1-\frac{R}{R_0}\right)^{-1} R'(2-\nu' r) + \nu' + \frac{4}{r} + e^\sigma\left(rR - \frac{4}{r}\right) + r\, e^\sigma \frac{16\pi\kappa}{c^2}\left(\rho_0 + \frac{p}{c^2}\right) \tag{25c}$$

$$\frac{p'}{c^2} = -\left(\frac{\nu'}{2}+\frac{2}{r}\right)\left(\rho_0 + \frac{p}{c^2}\right) + \frac{c^2}{8\pi\kappa}\left(1-\frac{R}{R_0}\right)^{n-2}\left\{-\frac{n-1}{R_0}e^{-\sigma}R'\left(\frac{\nu'}{r}-\frac{2}{r^2}\right)\right.$$
$$\left. -\left(1-\frac{R}{R_0}\right)\left[\frac{R}{r} + \frac{1}{r^2}(\nu' - 3\sigma')\,e^{-\sigma} + \frac{4}{r^3}(e^{-\sigma}-1)\right]\right\}. \tag{25d}$$

Equation (25a) loses its sense in the limit $R_0 \to \infty$, $R/R_0 \to 0$, as it contains terms proportional to R_0. The other equations in this limit reduce to the form which can be deduced from general relativity (Adler *et al* 1965).

As in § 3 we can make a (Laurent) series expansion for R, p, σ' and ν' at $r=0$. Asking for consistency with the equations (25) at $r=0$ and demanding that R and p should stay finite within the star, we get $R(0) = \text{constant} < R_0$ and $e^\sigma \to 1$, $e^\nu \to \text{constant}$ as $r \to 0$. As can be seen from equation (25d), $R \to R_0$ would result in a divergent pressure, since the curly bracket generally will not vanish.

When investigating the asymptotic behaviour of the solutions at infinity, we found that it is not possible to maintain the Schwarzschild solution ($R \equiv 0$) outside the matter, lest the curvature diverges to minus infinity inside the star. To give an argument for this behaviour, let us regard the contracted field equations (23)

$$-\frac{3}{\sqrt{-g}}\frac{\partial}{\partial r}\left(\sqrt{-g}\; e^{-\sigma} f^{(2)}(R)\frac{\partial}{\partial r}R\right) - R f^{(1)}(R) + 2 f(R) = \frac{8\pi\kappa}{c^2}T \tag{26}$$

at the edge r_0 of the mass distribution. If we assume $R/R_0 \ll 1$ outside the matter, then at $r=r_0$:

$$T \simeq \rho_0, \qquad f(R) \simeq R, \qquad f^{(1)}(R) \simeq 1, \qquad f^{(2)}(R) \simeq \tfrac{1}{3}a^2.$$

With $b^2 = a^2 e^{-\sigma(r_0)}$ equation (26) is approximately solved by:

$$R \approx \left(B + \frac{4\pi\kappa}{c^2}\rho_0 r_0\right)\frac{1}{r}\exp\left(\frac{r_0 - r}{b}\right)\theta(r - r_0)$$

$$+\left[\frac{B}{r}\exp\left(\frac{r_0 - r}{b}\right) + \frac{8\pi\kappa}{c^2}\rho_0 - \frac{4\pi\kappa}{c^2}\rho_0 r_0 \frac{1}{r}\exp\left(\frac{r - r_0}{b}\right)\right]\theta(r_0 - r) \tag{27}$$

where $\theta(r)$ is the step function.

This approximate solution certainly is not good for $r_0 \simeq 2m$, as we have here neglected terms containing the derivative of $e^{-\sigma}$ and $e^{-\nu}$, which in this limit surely will contribute. Postulating the Schwarzschild solution for $r \geq r_0$ determines the integration constant B to be $B = (4\pi\kappa/c^2)\rho_0 r_0$, and thus

$$R \approx \frac{8\pi\kappa}{c^2}\rho_0\left[1 - \frac{r_0}{r}\cosh\left(\frac{r - r_0}{6}\right)\right]\theta(r_0 - r) \tag{28}$$

is strongly negative for $r < r_0$. This means that in order to obtain reasonable solutions of (23) we have already to assume deviations from Schwarzschild outside the mass distribution. This is the point, where the empty space solutions shown in figures 1–4 become important.

We integrated the equations (25) by the method outlined in § 4, starting with the asymptotic behaviour (8), (9), (11) for $r \to \infty$, and determined $\alpha \neq 0$ numerically such that $p(0)$ and $R(0) < R_0$ were finite. From equation (27) one can expect that our solutions should correspond to $B = (4\pi\kappa/c^2)\rho_0 r_0 \exp(-2r_0/b)$ thus making R regular

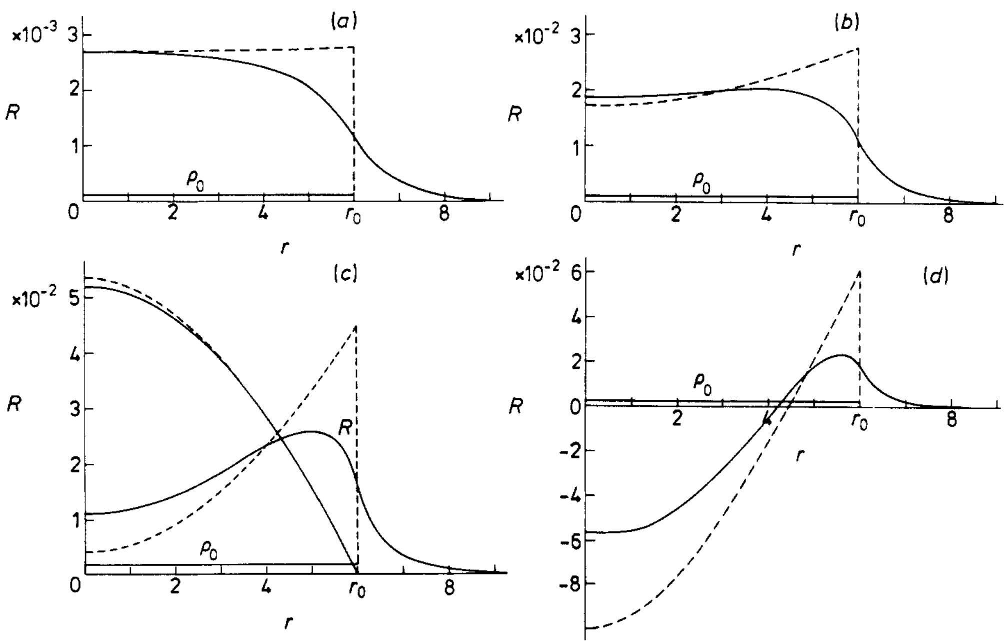

Figure 5. The curvature scalar $R(r)$ as a function of the mass of the star in the theory of Einstein (broken curves) and in the limiting curvature theory (full curves). Figure 5(c) also shows the pressure $p(r)$ in the two theories. (a) $m = 0{\cdot}1$, $r_0/2m = 30$; (b) $m = 1$, $r_0/2m = 3$; (c) $m = 1{\cdot}6$, $r_0/2m = 1{\cdot}875$; (d) $m = 2{\cdot}2$, $r_0/2m = 1{\cdot}364$.

at $r=0$. This means that $R(r_0) \simeq (4\pi\kappa/c^2)\rho_0[1+\exp(-2r_0/b)]$, i.e. about half the value one obtains with the Einstein equation at $r=r_0$. Our numerical results show that this is indeed true for $r_0 \gg 2m$ (see figure 5*a*), but that $R(r_0)$ becomes much less than $(4\pi\kappa/c^2)\rho_0$ when r_0 gets comparable to the Schwarzschild radius. Hence equation (27) becomes a poor estimate as $r_0 \to 2m$.

In figures 5 and 6 we present a series of solutions of the field equations (23) for variable ratios $r_0/2m$ and a constant limiting curvature $R_0=1.5$ $(a=1)$. The main feature of the curves is that the discontinuity of the curvature scalar R at $r=r_0$ is smeared out in the limiting curvature theory. At the same time $R_{\max} \equiv \max R(r)$ is less than the Einsteinian value $(8\pi\kappa/c^2)\rho_0$. In particular, while $R_{\max} \propto \rho_0$ in Einstein's theory, one observes a different behaviour of $R_{\max}(\rho_0)$ in the limiting curvature theory (figure 8): $R_{\max}$ decreases again as $r_0 \to 2m$. This can be understood as the effect of the terms in (26) containing the derivatives of R. Neglecting these terms in (26), the equation would be identical with the Einstein equation $R=(8\pi\kappa/c^2)T$ up to order $(R/R_0)^2$, which is small in all the solutions we obtained. So all the strong deviations

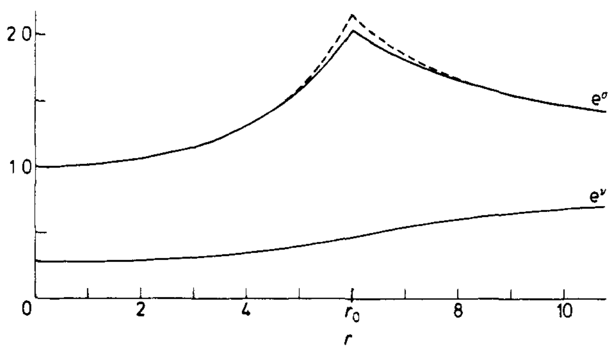

Figure 6. The metric components e^ν and e^σ in the limiting curvature theory (full curves) and in Einstein's theory (broken curve). e^ν is nearly identical in the two theories. $m=1{\cdot}6$, $r_0/2m=1{\cdot}875$.

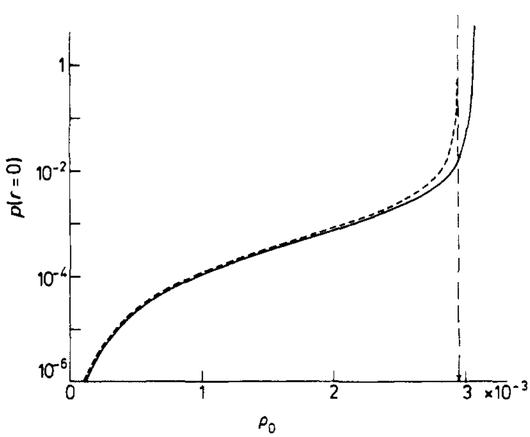

Figure 7. The pressure at the centre of the star, $p(r=0)$, as a function of the density of the star. In Einstein's theory (broken curve) $p(0)$ diverges at a density $\rho_0=2{\cdot}947\times10^{-3}$ (shown by arrow), corresponding to a radius $r=\frac{9}{8}(2m)$; in the limiting curvature theory (full curve) the divergence point lies at a density which is somewhat higher.

from Einstein's theory have their origin in the higher order terms in the differential equations, caused by the nonlinearities of the Lagrangian. These nonlinearities are responsible for the fact that R_{max} stays much below R_0 as long as R_0 does not become very small (see figure 8(*b*)). This means that the deviations in this theory from general relativity are not so much due to the limiting curvature R_0 built in, but to the higher order terms in the differential equations. The results will probably not be very different with any other nonlinear Lagrangian (see e.g. Bicknell 1974).

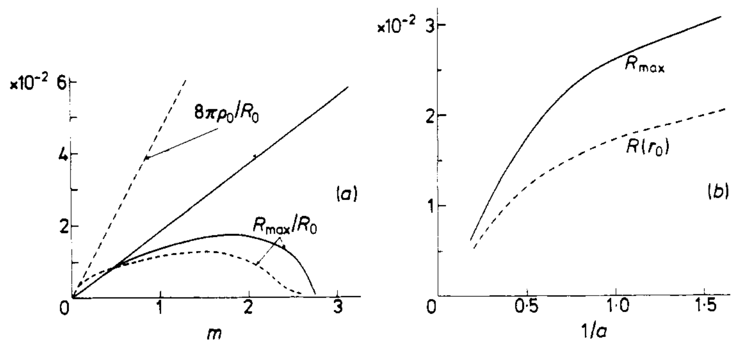

Figure 8. (*a*) The maximum value of the curvature, R_{max}/R_0, as a function of the mass of the star, compared to the Einsteinian value $8\pi\rho_0/R_0(\kappa = c = 1)$, for two values for the limiting curvature: $R_0 = 0{\cdot}06$ (broken curves) and $R_0 = 1{\cdot}5$ (full curves). One sees that deviations from Einstein's theory increase the smaller R_0 becomes. (*b*) R_{max} and $R(r_0)$ as a function of $1/a = R_0\sqrt{\frac{2}{3}}$. One realises the similar behaviour of the two quantities. For small values of a the dependence on $1/a$ is nearly linear. For (*a*) $r_0 = 6 =$ constant; for (*b*) $r_0 = 6$, $m = 1{\cdot}8$.

5. Consequences for gravitational collapse; event horizons in limiting curvature theories

Despite some strong deviations from Einstein's theory, one has to be aware of the fact that some very general features of the Einstein solutions are not destroyed in the limiting curvature theory. Figures 5 show the dependence of the solutions of (23) on the ratio $r_0/2m$. As in the Einstein case, a dip shows up in R at small radii which is due to the pressure and increases with the density of the star. The pressure itself behaves very similarly to that obtained from the theory of general relativity (figures 5(*c*) and 7); it diverges when $r_0/2m$ decreases below a critical value. In Einstein's theory this value is $r_0/2m = 9/8$; in our theory the numerical solutions cease to exist for an $r_0/2m$ ratio somewhat smaller than 9/8. This can be interpreted as due to the inclusion of nonlinearities into the Lagrangian which soften the equation of state of the star, but not enough to avoid divergent pressure completely.

The curvature in the centre of the star, $R(0)$, becomes negative when $r_0/2m \to 1$. In Einstein's theory this happens at $r_0/2m = 1{\cdot}8$; in our solutions the curvature $R(0)$ is always larger than the Einsteinian value (as a consequence of smearing out the discontinuity at $r = r_0$), therefore $R(0)$ becomes negative only for $r_0/2m < 1{\cdot}8$ (for $R_0 = 1{\cdot}5$, e.g. one finds a critical value $r_0/2m = 1{\cdot}665$). Thus the solutions of the limiting curvature theory in the limit $r_0/2m \to 1$ show a behaviour which is qualitatively very similar to that of the Einstein solutions. This statement is even supported by the additional fact, that (figure 9) $e^{\nu(r_0)}$ tends to zero as $r_0/2m \to 1$, i.e. that the gravitational redshift seems to become infinite when the star approaches its Schwarzschild radius. Furthermore, $R(r_0)$ tends to zero in the same limit (figure 8(*a*)), i.e. the solutions outside the star approach the Schwarzschild limit as $r_0/2m \to 1$. All these facts indicate that the stationary solution breaks down again when $r_0 \to 2m$, and that there again appears an event horizon. Indeed, for the perfect fluid model, we could not find any numerical solutions of (23) for $0 < r_0/2m < 1$ corresponding to the physical boundary conditions of a finite curvature and a finite pressure. Thus we are left with the same situation as in Einstein's theory: when the mass (by any mechanism) collapses or is pressed through its Schwarzschild radius, *stationary physical solutions of the field equations for* $0 < r_0/2m < 1$ *do not exist.*

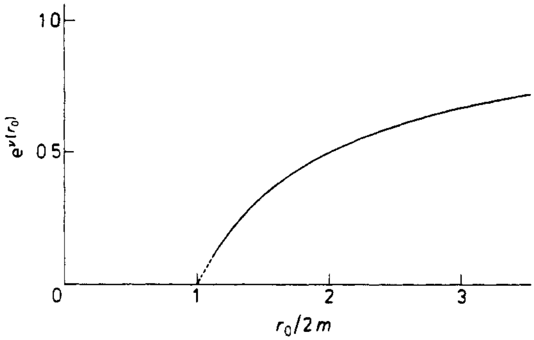

Figure 9. The metric component e^{ν} at the surface of the star as a function of $r_0/2m$. $r_0 = 6 =$ constant, $R_0 = 1{\cdot}5$.

This means that within the usual hydrodynamical model for the matter the gravitational collapse cannot be avoided. However, as in this theory the radial coordinate does not become time-like for $r < 2m$ (in contrast to Einstein's theory), if one assumes the solutions of figures 1–4 to be the solutions for $r > r_0$, *there does not occur an event horizon when the star collapses through its Schwarzschild radius.* In Einstein's theory any distribution of matter with $r_0 < 2m$ is inevitably forced to collapse to a point, because the r-coordinate is time-like. In our theory this is not the case, and at first sight one might hope to be able to construct another equation of state which can prevent the collapse of the star to a point singularity. It seems that an equation of state allowing for an anisotropic pressure can avoid the pathological behaviour of the solutions in the limit $r_0 \to 2m$. For a further discussion of this question see the next section and Heinz *et al* (1978), where we shall carefully investigate the structure of the source T_i^k.

6. Conclusions

The main results of the investigations presented in this paper may be summarised as follows: One cannot learn very much from the empty space solutions of the generalised field equations alone; in addition to the well known solutions for euclidean space-time ($e^{\nu} \equiv e^{\sigma} \equiv 1$) and for the boundary conditions for a point mass (Schwarzschild solution, $e^{\nu} = e^{-\sigma} = 1 - (2m/r)$, see comment in the appendix) one finds one solution corresponding to a globally constant curvature $R \equiv \frac{8}{9}R_0$ (for the theory with $n = +1/2$), which might possibly be of interest as a cosmological solution for a constantly curved universe (to be compared with the solution of the Einstein equations for empty space including a cosmological term $\Lambda = \frac{2}{9}R_0$) and a discrete spectrum of solutions with Schwarzschild limit at infinity characterised by the fact that at the origin R tends to the limiting curvature R_0 and the metric components e^{ν}, e^{σ} vanish at $r = 0$. The physical significance of the last type of solution only becomes clear in context with the solutions for an extended liquid drop, and it is not easy to find reasonable physical arguments to select one solution from this discrete spectrum. In order to make the empty space solution unique, we imposed the condition that the velocity of light $c = \exp[\frac{1}{2}(\nu - \sigma)]$ should stay finite and greater than zero everywhere in space–time.

Investigations of the solutions for an extended liquid drop show that the most important modifications of Einstein's solution are not due to the fact that we introduced a limiting curvature into the theory, but are caused by the higher order terms in the differential equations coming from the nonlinearities in the Lagrangian. Because of these higher order terms the discontinuity of the curvature scalar at the edge of the mass distribution is smeared out, and we no longer have the Schwarzschild solution ($R(r) = 0$ for $r \neq 0$) outside the matter, but an exponentially decreasing curvature corresponding to the last type of empty space solutions discussed above. The width of this smearing-out effect is given by the limiting curvature; it is proportional to $a = [3(1 - n/R_0^2]^{1/2}$. This means that in a range of the order of $[3(1 - n)/R_0^2]^{1/2}$ above the mass distribution deviations from the $1/r^2$ behaviour of the gravitational force should occur. As the $1/r^2$ law for gravitation has been verified very exactly for large distances, but not so clearly at laboratory dimensions (Long 1974), $a = [3(1 - n)/R_0^2]^{1/2}$ should not be greater than a few centimetres. This means that the limiting curvature must be larger than about $1\ \mathrm{cm}^{-2}$, which corresponds to the value caused by a matter density of $\rho_0 \sim 10^{27}\ \mathrm{g\ cm}^{-3}$ in Einstein's theory.

The most important result, however, is that within the perfect fluid model the limiting curvature theory cannot avoid the collapse to a point singularity, although we tried to get rid of this behaviour from the beginning by introducing a limiting curvature and thus forbidding the occurrence of infinite curvatures. In this limiting curvature theory, too, the stationary solutions cease to exist, when the radius of the star approaches its Schwarzschild radius. The gravitational pressure becomes infinite and cannot be balanced by the inner pressure of the liquid drop, and the star thus is forced to collapse to one of the point-source solutions discussed in § 3.

This result naturally leads to the question why there is no continuous transition possible from the solutions for an extended liquid drop to the point-source solutions of figures 1–4. The reason for this impossibility is the same in Einstein's theory (Heinz *et al* 1978): inserting the Schwarzschild solution back into the Einstein equations gives

$$T_i^k = (M/4\pi r^2)\,\mathrm{diag}\,(\delta(r), \delta(r), \tfrac{1}{2}r\delta'(r), \tfrac{1}{2}r\delta'(r)) \tag{29}$$

where M is the mass of the star. This shows that for the point source corresponding to the Schwarzschild solution (which can be shown (Heinz *et al* 1978) to be the only possible point source with vanishing four-divergence in Einstein's theory) the T_0^0 and T_1^1 components are equal to each other, and the three spatial components differ from each other, i.e. the pressure of the source is anisotropic. In other words, the Schwarzschild point source cannot be obtained as the limit of a series of static perfect-fluid models.

Similarly one obtains for the solutions for the limiting curvature theory shown in figure 1 (with $n = 1/2$) (see appendix):

$$\frac{\kappa}{c^2} T_i^k \exp\left(\frac{\nu+\sigma}{2}\right) = c(0)\frac{\rho}{2}\sqrt{\left(\frac{2R_0}{4A}\right)}\frac{\delta(r)}{4\pi r^2} \times \begin{pmatrix} \frac{1}{2}+\frac{1}{4}(2\nu_{-1}-\sigma_{-1}) & & & \\ & -1-\frac{1}{4}\nu_{-1} & & \\ & & 1+\frac{1}{4}(\nu_{-1}-\sigma_{-1}) & \\ & & & 1+\frac{1}{4}(\nu_{-1}-\sigma_{-1}) \end{pmatrix} \tag{30}$$

where $c(0)$ is the velocity of light at $r=0$, $\rho = 2+\nu_{-1}-\sigma_{-1}$, ν_{-1} and σ_{-1} are the coefficients of the leading terms in $\nu'(r)$ and $\sigma'(r)$ near $r=0$ (see equation (16)) and A is defined by the asymptotic expansion for R (cf. equation (17))

$$R(r) = R_0 - Ar^\rho \qquad (r \text{ small}). \tag{31}$$

We did not write down here the tensor T_i^k, but the tensor density $T_i^k \exp[\frac{1}{2}(\nu+\sigma)]$, as the only quantity of physical interest is

$$\int T_i^k \sqrt{-g}\, d^4x = \int T_i^k \exp\left(\frac{\nu+\sigma}{2}\right) dt\, d^3r.$$

For the simplest case $\nu_{-1} = \sigma_{-1} = 2$ (corresponding to a finite velocity of light everywhere) equation (30) leads to

$$\frac{\kappa}{c^2} T_i^k \exp\left(\frac{\nu+\sigma}{2}\right) = c(0)\left(\frac{2R_0}{R''(0)}\right)^{1/2} \frac{\delta(r)}{4\pi r^2} \operatorname{diag}(1, -\tfrac{3}{2}, 1, 1). \tag{32}$$

The numerical calculations show that the factor $(2R_0/R''(0))^{1/2}$ is exactly proportional to the mass of the star (see figure 10). Thus the point source for the limiting curvature theory again has nothing to do with the hydrodynamical model for the perfect fluid, and the impossibility of a continuous transition from the solutions for an extended liquid drop to the point-source solutions becomes obvious. An investigation of generalised models for the matter tensor which allow for the limit (32) is in progress.

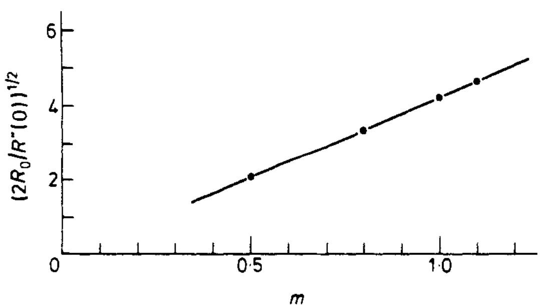

Figure 10. $(2R_0/R''(0))^{1/2}$ as a function of m for the point-source solutions of § 3, showing strict proportionality.

Acknowledgments

We wish to thank Dr J Rafelski, Dr H Ruck and M Soffel for many fruitful discussions.

Note added in proof. In order to investigate the stability of stars against gravitational collapse in limiting curvature theories, we have looked for stable orbits of test particles in the metric shown in figure 1. These are given by the minima of the effective potential $V_{\text{eff}}^2 = \exp(\nu(r))(m_{\text{p}}^2 + l^2/r^2)$ where m_{p} is the mass of the test particle and l its angular momentum. The resulting radii of stable orbits are connected with l by $l^2 = r^2[2/(2-\nu' r)-1]$. For the Schwarschild metric the smallest stable orbit has the radius $r_{\min} = 6m$ (where m is the central mass), which is the reason for instability against collapse in Einstein's theory when the star radius approaches the critical value $r_0 \sim 6m$ (Oppenheimer and Volkoff 1939). For the limiting curvature theory $r_{\min}$ turns out to be larger than $6m$, approaching infinity as the limiting curvature R_0 goes to zero. Thus in our theory stars will become unstable against gravitational collapse earlier than in general relativity. We conclude that the introduction of a limiting curvature scalar does not help to avoid the final collapse of a heavy star into a point singularity.

Appendix

We can write the equation for T_0^0 as

$$\frac{8\pi\kappa}{c^2} T_0^0 = f^{(3)}(R) R'^2 g^{11} + f^{(2)}(R) g^{11}\left[R'' + R'\left(\frac{2}{r} - \frac{\sigma'}{2}\right)\right] - f^{(1)}(R) R_0^0 + \frac{1}{2} f(R) \tag{A1}$$

where $f^{(n)}(R)$ is the nth derivative of f with respect to R and the prime denotes differentiation with respect to r. (A1) can be transformed to

$$\begin{aligned}\frac{8\pi\kappa}{c^2} T_0^0\, \mathrm{e}^{(\nu+\sigma)/2}\\ &= -\mathrm{e}^{\nu/2} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2\, \mathrm{e}^{-\sigma/2} \frac{\partial}{\partial r} f^{(1)}\right) - \mathrm{e}^{(\nu+\sigma)/2} f^{(1)} \frac{1}{r^2} \frac{\partial}{\partial r}(r(\mathrm{e}^{-\sigma} - 1))\\ &\quad + \tfrac{1}{2}\, \mathrm{e}^{(\nu+\sigma)/2} f - \tfrac{1}{2} R\, \mathrm{e}^{(\nu+\sigma)/2} f^{(1)}.\end{aligned} \tag{A2}$$

Inserting the expansions (12a), (12b), (17), we get in lowest order (higher order terms don't give δ-function contributions):

$$\begin{aligned}\frac{8\pi\kappa}{c^2} T_0^0\, \mathrm{e}^{(\nu+\sigma)/2}\left(\frac{A}{R_0}\right)^{1-n} \mathrm{e}^{(\bar{\sigma}-\bar{\nu})/2}\\ &= -r^{-\nu_{-1}/2-2} \frac{\partial}{\partial r} r^{2-\sigma_{-1}/2} \frac{\partial}{\partial r} r^{(n-1)\rho} - r^{(\nu_{-1}+\sigma_{-1})/2-2+(n-1)\rho} \frac{\partial}{\partial r} r^{1-\sigma_{-1}}.\end{aligned} \tag{A3}$$

Using the identities (for $\sigma_{-1} \geq 2$)

$$1 = 2 + \frac{\nu_{-1} + \sigma_{-1}}{2} + (n-1)\rho \tag{A4}$$

and

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^{2+(\nu_{-1}-\sigma_{-1})/2}\frac{\partial}{\partial r}r^{(n-1)\rho}=(n-1)\rho\frac{\delta(r)}{r^2} \tag{A5}$$

(as can be verified by integrating with a test function), we get

$$\begin{aligned}\frac{8\pi\kappa}{c^2}T_0^0\,\mathrm{e}^{(\nu+\sigma)/2}\left(\frac{A}{R_0}\right)^{1-n}\mathrm{e}^{(\bar\sigma-\bar\nu)/2}\\ =-(n-1)\rho\frac{\delta(r)}{r^2}+\frac{\nu_{-1}}{2}\frac{1}{r^2}\frac{\partial}{\partial r}r^{1+(\nu_{-1}-\sigma_{-1})/2}r^{(n-1)\rho}\\ -\left[\frac{\nu_{-1}}{2}\left(1+\frac{\nu_{-1}-\sigma_{-1}}{2}\right)+1-\sigma_{-1}\right]r^{(\nu_{-1}-\sigma_{-1})/2-2}r^{(n-1)\rho}.\end{aligned} \tag{A6}$$

From (16) one can show that the last term vanishes identically. The second term is (integration with a test function!)

$$\frac{\nu_{-1}}{2}\frac{1}{r^2}\frac{\partial}{\partial r}(r^{-1+(\nu_{-1}-\sigma_{-1})/2+(n-1)\rho})=\frac{\nu_{-1}}{2}\frac{\delta(r)}{r^2}. \tag{A7}$$

Thus one has with (A4)

$$\begin{aligned}\frac{8\pi\kappa}{c^2}T_0^0\,\mathrm{e}^{(\nu+\sigma)/2}\\ =\left(-(n-1)\rho+\frac{\nu_{-1}}{2}\right)\left(\frac{A}{R_0}\right)^{n-1}\mathrm{e}^{(\bar\nu-\bar\sigma)/2}\frac{\delta(r)}{r^2}\\ =\left(\frac{1}{2}+\frac{2\nu_{-1}-\sigma_{-1}}{4}\right)\left(\frac{A}{R_0}\right)^{n-1}\mathrm{e}^{(\bar\nu-\bar\sigma)/2}2\frac{\delta(r)}{r^2}.\end{aligned} \tag{A8}$$

The other components of T_i^k may be developed similarly. One obtains from equation (23)

$$\begin{aligned}\frac{8\pi\kappa}{c^2}T_1^1\,\mathrm{e}^{(\nu+\sigma)/2}\\ =-\mathrm{e}^{(\nu-\sigma)/2}\left(\frac{\nu'}{2}+\frac{2}{r}\right)\frac{1}{r^2}\frac{\partial}{\partial r}r^2f^{(1)}+\mathrm{e}^{(\nu+\sigma)/2}\frac{3\,\mathrm{e}^{-\sigma}+1}{r^2}f^{(1)}\\ -\tfrac{1}{2}R\,\mathrm{e}^{(\nu+\sigma)/2}f^{(1)}+\tfrac{1}{2}\mathrm{e}^{(\nu+\sigma)/2}f,\end{aligned} \tag{A9}$$

$$\begin{aligned}\frac{8\pi\kappa}{c^2}T_2^2\,\mathrm{e}^{(\nu+\sigma)/2}\\ =\frac{8\pi\kappa}{c^2}T_3^3\,\mathrm{e}^{(\nu+\sigma)/2}\\ =-\frac{\partial}{\partial r}r\,\mathrm{e}^{(\nu-\sigma)/2}\frac{\partial}{\partial r}\frac{f^{(1)}}{r}-\mathrm{e}^{(\nu-\sigma)/2}\frac{2}{r}\frac{\partial}{\partial r}f^{(1)}+\frac{2\,\mathrm{e}^{-\sigma}-1}{r^2}\mathrm{e}^{(\nu+\sigma)/2}f^{(1)}\\ +\tfrac{1}{2}f\,\mathrm{e}^{(\nu+\sigma)/2}.\end{aligned} \tag{A10}$$

Inserting again the expansions (12*a*), (12*b*), (17) one realises that there occur only terms of the same structure as already discussed above. Defining

$$
\begin{aligned}
X &\equiv -\frac{1}{8\pi}\frac{1}{r^2}\frac{\partial}{\partial r}r^{(\nu_{-1}-\sigma_{-1})/2}\frac{\partial}{\partial r}r^{(n-1)\rho}=\left(\frac{1}{2}+\frac{\nu_{-1}-\sigma_{-1}}{4}\right)\frac{\delta(r)}{4\pi r^2} \\
Y &\equiv \frac{1}{8\pi}\frac{1}{r^2}\frac{\partial}{\partial r}r^{1+(\nu_{-1}-\sigma_{-1})/2}r^{(n-1)\rho}=\frac{1}{2}\frac{\delta(r)}{4\pi r^2}
\end{aligned}
\tag{A11}
$$

one finds the following structure of T_i^k:

$$
\frac{\kappa}{c^2}T_i^k\,\mathrm{e}^{(\nu+\sigma)/2}=\left(\frac{A}{R_0}\right)^{n-1}\mathrm{e}^{(\bar\nu-\bar\sigma)/2}\left[X\,\mathrm{diag}(1,0,1,1)+Y\,\mathrm{diag}\left(\frac{\nu_{-1}}{2},-2-\frac{\nu_{-1}}{2},1,1\right)\right]
\tag{A12}
$$

and

$$
\frac{\kappa}{c^2}T\,\mathrm{e}^{(\nu+\sigma)/2}=3\left(\frac{A}{R_0}\right)^{n-1}\mathrm{e}^{(\bar\nu-\bar\sigma)/2}X.
\tag{A13}
$$

In conclusion we mention that inserting the Schwarzschild solution $\mathrm{e}^{\nu}=\mathrm{e}^{-\sigma}=1-2m/r$ into the field equations leads to a T_i^k tensor which is composed not only of δ functions, but also of derivatives and powers of δ functions. This does not seem to be a sensible choice for the source of a gravitational field. *Therefore the Schwarschild solution cannot be regarded as a true solution of the field equations for the limiting curvature theory.*

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